Lecture 6-1

More on Formation Control

Today:

- Formation Control true navigation functions & network tension
- project

Quote of the day

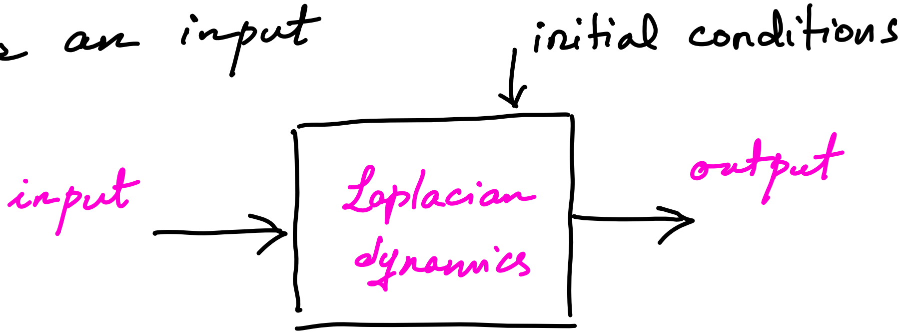
"My mom says: 'Why aren't you a doctor?' and I'm like, 'I am a doctor!' & she's all, 'No, I mean a real doctor.' She reads my books, but she says they give her a headache."

Brian Greene

Last time we looked at formation control with RSS

relative state specifications

& ended up with an input/output dynamics that has an embeded "Laplacian" dynamics that also has an input



creates coordination among the nodes while also doing something useful like making a formation

In the meantime, the network itself can be setup in such a way that it only responds to an initial condition, and intrinsincly accomplishes the given task ... the key requirement is that information needed for each agent to act is <u>local</u> ... no centralization in information sharing.

Today we want to look at intrinsically modifying the network to achieve our goal, & we will do this in the context of mobile robots.

The canonical model that we end up using is of the form

$$\dot{x} = -\mathcal{L}_{w(x)}(x)$$

where $w(x)$ are state-dependent edge weights. Note that generally a weighted Laplacian looks like

$$\mathcal{L}_w = \underbrace{D(G)}_{n\times m} \underbrace{W}_{m\times m} \underbrace{D(G)^T}_{m\times n}$$

$\hookrightarrow$ weights on the edges of the network with incidence matrix $D(G)$.

So essentially we will explore useful dynamics for a group of robots via the state-dependent nonlinear consensus:

$$\dot{x} = -D(G)\,W(x)\,D(G)^T x$$

But this is the end of the story! Let us start from the begining.

Once upon a time there was a group of n robots $\{1, 2, \ldots, n\}$ &

$$ij \in E \iff \|x_i - x_j\| \leq \delta$$

in $\mathbb{R}^p$

Denote by $z_{ij} = x_j - x_i$. Consider the set

$$D_{G,\delta} = \left\{ x \in \mathbb{R}^{pn} \;\middle|\; \|z_{ij}\| \leqslant \delta \,,\; \forall\, i,j \right\}$$

all realizations that lead to a complete graph

It also becomes convenient to introduce an ϵ-interior of this set, namely

$$D^{\epsilon}_{G,\delta} = \left\{ x \in \mathbb{R}^{pn} \;\middle|\; \|z_{ij}\| \leqslant \delta - \epsilon \quad \forall\, ij \right\}$$

We define the edge tension as

$$V_{ij}(\delta, x) = \begin{cases} \dfrac{\|z_{ij}\|^2}{\delta - \|z_{ij}\|} & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

& the total network tension as

$$V(\delta, x) = \frac{1}{2} \sum_{i,j} V_{ij}(\delta, x)$$

you notice that if $\|z_{ij}\| \rightarrow \delta$, the $V_{ij}(\delta, x) \uparrow$

To prevent this we look an algorithm where the network tension does not grow. Well, if we don't want a scalar function to grow we can go in the direction of the negative gradient. What is this direction?

First notice that

$$\frac{\partial V_{ij}}{\partial x_i} = \begin{cases} \dfrac{2\delta - \|z_{ij}\|}{\left(\delta - \|z_{ij}\|\right)^2} & \text{if } ij \in E \\ 0 & \text{otherwise} \end{cases}$$

So it would make sense to let

$$-\sum_{j\sim i} \frac{2\delta - \|z_{ij}\|}{(\delta - \|z_{ij}\|)^2}(x_i - x_j)$$

$$\dot{x}_i = -\sum_{j\sim i} \frac{\partial V_{ij}}{\partial x_i} = -\nabla_{x_i} V(\delta, x)$$

The potential problem is that $\|z_{ij}\| \to \delta$ & the gradient becomes ill-defined. However we can show that this cannot happen:

Lemma: Let $x_0 \in \mathcal{D}^{\epsilon}_{G,\delta}$ for some $\epsilon \in (0,\delta)$, where G is connected. Then the set

$$\Omega(\delta, x_0) = \{x \mid V(\delta, x) \leq V(\delta, x_0)\}$$

is invariant if $\dot{x}_i = -\nabla_{x_i} V(\delta, x)$

The main part of the proof involves showing that if $V(\delta, x)$ is not increasing, we can't have $\|z_{ij}\| \to \delta$ for some pair ij. In order to show this we can consider

$$V_\epsilon^{max} = \max_{x \in D^\epsilon_{G,\delta}} V(\delta, x) \quad \Big\} \text{ max network tension}$$

every pair is at $\delta - \epsilon$ distance

\# of edges

$$\Rightarrow V_\epsilon^{max} = \frac{m(\delta-\epsilon)^2}{\epsilon}$$

Suppose all this tension is from a single edge, that is

$$V_\epsilon^{max} = \frac{\|z_\epsilon\|^2}{\delta - \|z_\epsilon\|}$$

$$\Rightarrow \frac{m(\delta-\epsilon)^2}{\epsilon} = \frac{\|z_\epsilon\|^2}{\delta - \|z_\epsilon\|} \Rightarrow \|z_\epsilon\| \leq \delta - \frac{\epsilon}{m} < \delta$$

So even in the case that all the network tension is concentrated in one edge, still $\|Z\| < \delta$.

If this set is invariant, LaSalle's Principle now implies that $x(t) \to$ largest invariant set contained in $\{x \mid \dot{\mathcal{V}} = 0\}$

This however follows from the realization that the dynamics is really

$$\dot{x} = -D(G)\, W(x)\, D(G)^T x$$

& $W(x)$ has positive diagonal elements bounded away from zero

However in most practical applications, robots will move around & the graph will not remain static (SIG vs. DIG)

static interaction graph

dynamic interaction graph

The main problem is that we cannot allow new "tensions" to arise when an agent has a new neighbor arbitrary.

$$\dot{x}_i = \begin{cases} -\dfrac{\partial V_{ij}}{\partial x_i} & \sigma_{ij} = 1 \\ \text{otherwise} & 0 \end{cases}$$

& we let

$$\sigma_{ij}[t^+] = \begin{cases} 0 & \text{if } \sigma_{ij}[t^-] = 0 \ \& \\ & \|z_{ij}\| > \delta - \epsilon \\ 1 & \text{otherwise} \end{cases}$$

The graph induced by σ is denoted by G_σ : note that this is <u>not</u> just induced by the <u>geometry</u>

Thm: let G_0 be the initial graph & $x_0 \in D^{\epsilon}_{G^0,\delta}$ for $0<\epsilon<\delta$. ; & $G_{0,\sigma}$ is connected. Then $x(t) \to \mathcal{A}$.

This setup can now be extended to formation control problems.

Suppose our desired formation is such that

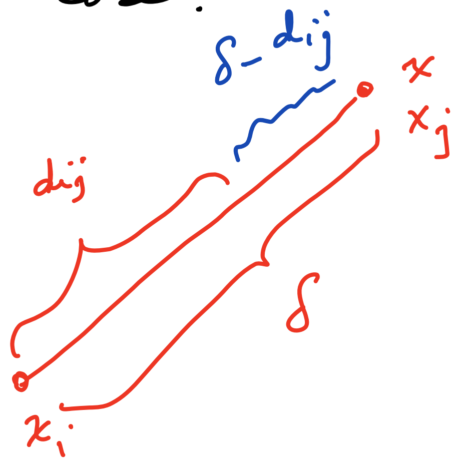
$$\|z_{ij}\| \to \|d_{ij}\| < \delta$$

information exchange threshold

What should our tension be in this case?

Let
$$\begin{cases} d_{ij} = \tau_i - \tau_j & \text{for } ij \in E_d \\ y_i = x_i - \tau_i \\ z_{ij} = x_i - x_j \end{cases}$$

$$\Rightarrow \quad y_i - y_j = z_{ij} - d_{ij}$$



$$V_{ij} = \begin{cases} \dfrac{\| z_{ij} - d_{ij} \|^2}{\delta - \| d_{ij} \| - \| z_{ij} - d_{ij} \|} & \text{if } ij \in E \text{ spanning tree} \\ 0 & \text{otherwise} \end{cases}$$

The gradient flow then looks like

$$\dot{x}_i = - \sum_{j \sim i} \frac{2(\delta - \| d_{ij} \| - \| z_{ij} - d_{ij} \|)}{(\delta - \| d_{ij} \| - \| z_{ij} - d_{ij} \|)^2} (x_i - x_j - d_{ij})$$

✓ invariance

✓ Converges $\quad \| x_i - x_j \| \to \| d_{ij} \|$

✓ Network Integrity: $\| x_i - x_j \| < \delta$