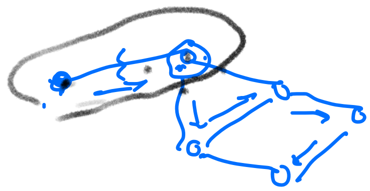
Lecture 2-2

Today:
- graph & matrices
- spectral graph theory
- consensus alg.

Connectivity



$\kappa_0(G) = 1$

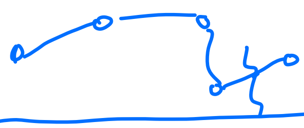
$\kappa_1(G) = 1.$

node connectivity: min # of nodes to remove before the graph becomes disconnected.

$\kappa_0(G)$

min # of edges

$\kappa_1(G)$

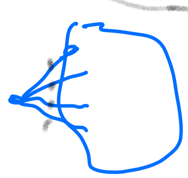


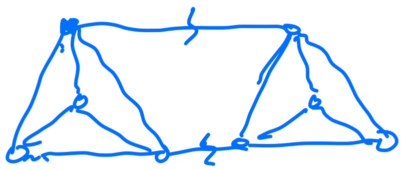
max $\kappa_0(G)$

G constrained

$\rightsquigarrow$ $\kappa_0(G) \le \kappa_1(G) \le \delta(G)$

min deg of nodes in G

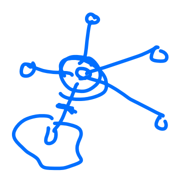




$\kappa_0(G) = 2$

$\kappa_1(G) = 2$

$\delta(G) = 3$



$G = (V, E)$

$A(G)$ = adjacency matrix

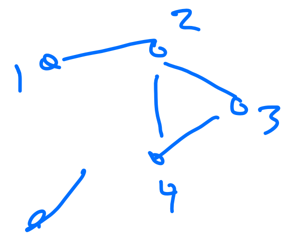
$$[A(G)]_{ij} = \begin{cases} 1 & i \sim j \\ 0 & \text{otherwise} \end{cases}$$

$ij \in E(G)$

$j \in N(i)$ $\quad i \in N(j)$

$$A(G) = \begin{array}{c} \\ 1 \end{array}\begin{array}{c} \quad 2 \\ \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix} \end{array}$$

$n \times n$



$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$ symmetric

trace $A = 0 = \sum$ eigenvalues of A

sum

$\cancel{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$

$A x = \lambda x$

$n \times n$ $\quad n \times 1$

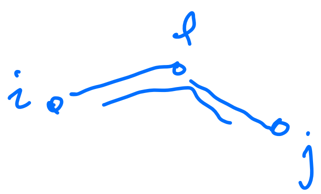
$\dot{x} = A(G)\, x$

↪ unstable dynamics.

$[A(G)^2]_{ij}$

$$[A\,B]_{ij} = \sum_{l} [A]_{il} [B]_{lj}$$

$n \times m$ $\quad m \times k$



$$[A(G)^2]_{ij} = \sum_{l} A_{il} A_{lj}$$

$[A(G)^2]_{ij}$ = # walks of length 2 btw i and j

$[\underbrace{A(G)^k}]_{ij}$ = # of walks of length k btw i & j

dist. btw i & j = $\min\limits_{k} \underbrace{[A(G)^k]}_{ij} \neq 0$

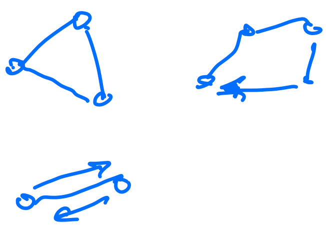
Adjacency spectrum: $\lambda_1, \lambda_2, \ldots, \lambda_n$

real #'s

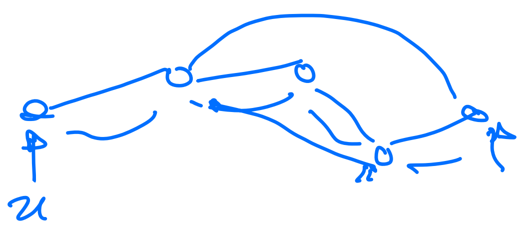
because of symmetry of $A(G)$.

$$\sum_{i=1}^{n} \lambda_i^k(A(G)) = \text{trace } A(G)^k$$

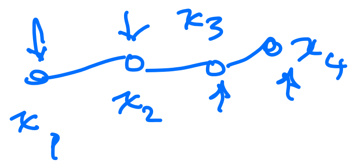
$$= \sum_i [A(G)^k]_{ii}$$



= counts the # of closed walks for all the nodes



$$\dot{x} = \underbrace{A(\mathcal{G})}_{n \times n} x + \underbrace{B}_{n \times p} u.$$

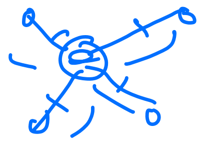


$\dot{x}_1 = x_2$

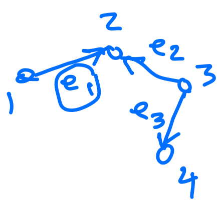
$\dot{x}_2 = x_3$

$\dot{x}_3 = x_4$

? min # nodes to inf. to make this sys controllable / stabilize.

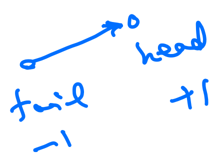


$\underbrace{D(\mathcal{G})}_{n \times m}$ = incidence matrix



orient e's.

$$D(\mathcal{G}) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$



Laplacian: $\underbrace{L(\mathcal{G})} = \underbrace{D(\mathcal{G})}\,\underbrace{D(\mathcal{G})}^T$

$n \times n$ $\quad n \times m$ $\quad m \times n$

$\rightarrow$ $\mathcal{L}(G)$ is positive semi definite

$x^T \mathcal{L}(G)\, x \geq 0 \quad \forall x$

$\lambda_1(\mathcal{L}(G))$ $\quad \lambda_2(\mathcal{L}(G))$ $\quad \ldots \quad$ $\lambda_n(\mathcal{L}(G))$

≥ 0 $\quad \geq 0$ $\quad \geq 0$

$\dot{x} = -\mathcal{L}(G)\, x \longrightarrow \max \lambda_2(G)$

const. G_s.