Lecture 3-1

Jan. 17, 2024

Today:

- Consensus algorithm
- Graph Spectra/consequences

Quote of the day:

"Whenever people agree with me, I always feel I must be wrong."

Oscar Wilde

Graphs $\longleftrightarrow$ $\underbrace{\text{Matrices}}$

$\downarrow$

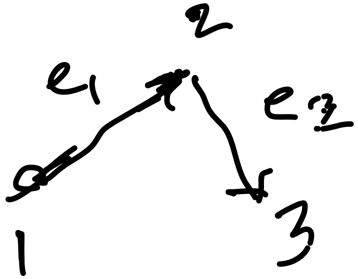
Adjancency

Incidence $\longrightarrow$ $\underbrace{D(G^{\sigma})}_{n \times m}$ orientation

Laplacian

$$\mathcal{L}(G) = \underbrace{\underbrace{D(G)}_{n \times m}\,\underbrace{D(G)^T}_{m \times n}}_{n \times n}$$

\# of edges



$$\begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$D(G^{\sigma})D(G^{\sigma})^{T} = \mathcal{L}(G) = \boxed{\Delta(G) - A(G)}$$

degree matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1) $\mathcal{L}(G)$ is p.s.d

$$x^T \mathcal{L}(G) x \geq 0 \quad \forall x$$

$$x^T D D^T x = (D^T x)^T (D^T x)$$

$$= \| D^T x \|^2 \geq 0 \quad \forall x$$

Laplacian Spectrum:

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

? $$\mathcal{L}(G) = \Delta(G) - A(G)$$

$$\mathcal{L}(G)\,\mathbb{1} = 0 \implies \mathbb{1} \in N(\mathcal{L})$$

null space

space / Kernel.

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

λ_1

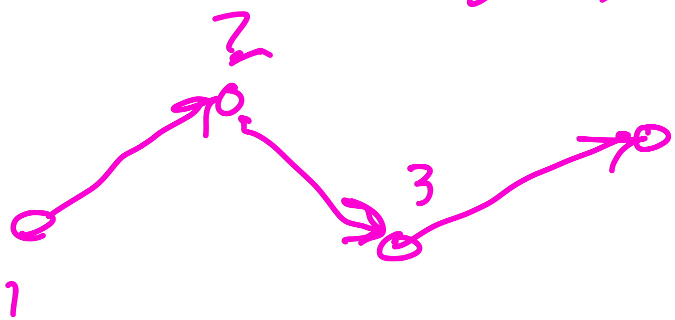
$$\mathcal{L}(G)\,\mathbb{1} = 0 \cdot \mathbb{1}.$$

? $\lambda_2 > 0$?

$$\underbrace{\underset{n \times n}{\mathcal{L}(G)}\ \overset{n \times 1}{v} = 0}$$

$$\underbrace{v^T D D^T v = 0} \longrightarrow \underline{D^T v = 0}$$

$$v^T D = 0$$



$$\underbrace{D = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}}$$

$$\underbrace{\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = 0$$

$$(\alpha \mathbb{1})^T = \begin{bmatrix} \alpha & \alpha & \alpha \end{bmatrix}$$

If G is connected the only v for which $\mathcal{L}(G)v = 0$ is when $v = \alpha \mathbb{1}$ for some α.

$\Rightarrow$ $\lambda_2(\mathcal{L}) > 0$ iff G is connected

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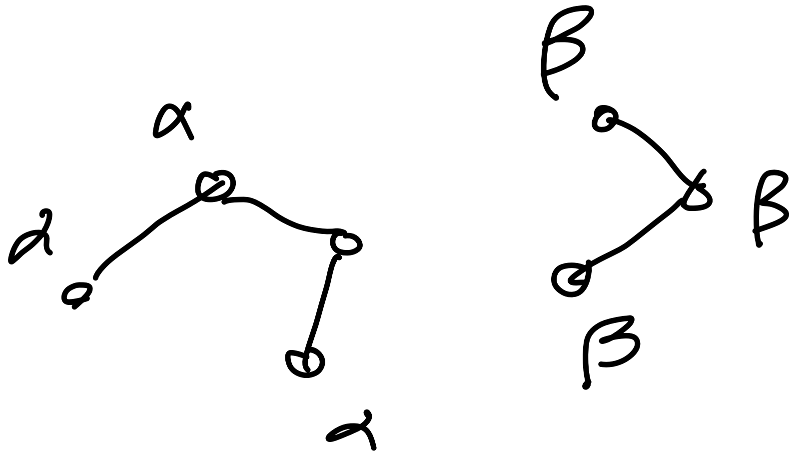
$$\mathcal{L}(G)v = 0$$



$\to$ $\dim \mathcal{N}(\mathcal{L}) = c = \#$ of connected components

$\to$ $\operatorname{rank} \mathcal{L}(G) = n - c$

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$
$$\lambda_1 = 0$$

$\lambda_2 > 0$ for a connected graph.

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show

$\to$ $\lambda_2 \leq \kappa_0(G) \leq \kappa_1(G)$

$\hookrightarrow$ algebraic connectivity.

Courant-Fisher: a symmetric $A \in \mathbb{R}^{n \times n}$

$$\min_{\|x\|=1} \boxed{\underbrace{x^T A x}_{f(x)}} = \lambda_1(A)$$

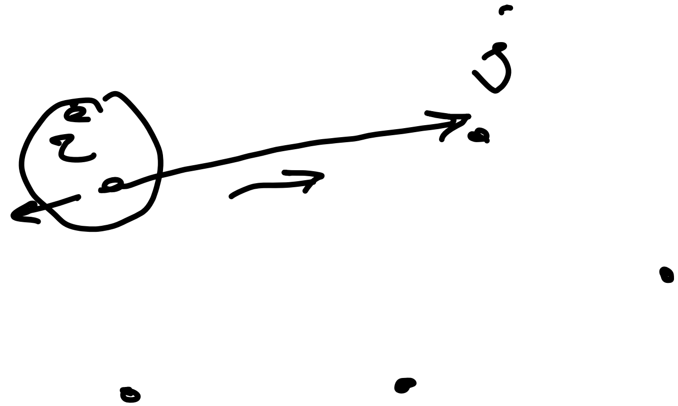
$$\max_{\|x\|=1} x^T A x = \lambda_n(A)$$

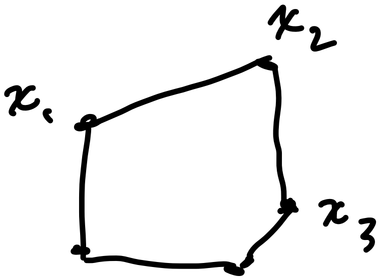
$$\min_{\substack{\|x\|=1 \\ x \perp v}} x^T A x = \lambda_2(A)$$

$x \perp v$ ← eigenvector corresponding to $\lambda_1(A)$

$$\dot{x}_i = x_j - x_i$$

$$\dot{x}_j = x_i - x_j$$





$$\begin{bmatrix} \vdots \\ \dot{x}_i \\ \vdots \end{bmatrix} = \sum_{j\sim N(i)} w_{ij}\,(x_j - x_i)$$

deg of node

$$\dot{x}_i = -d_i x_i - \underbrace{(1)\, x_j}_{\text{neighbor of } i}$$

$$\boxed{\dot{x} = -L(G)\,x}$$

$x(t) \rightarrow\ ?$

$$\mathcal{L}(G) = U \Lambda U^T$$

$$\begin{bmatrix} | & | & & | \\ \frac{\mathbb{1}}{\sqrt{n}} & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \ddots \; \lambda_n \end{bmatrix} \begin{bmatrix} \mathbb{1}^T/\sqrt{n} \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$x(t) = e^{-\mathcal{L}(G)t} x_0$$

$$= U e^{-\Lambda t} U^T x_0$$

$$= U \begin{bmatrix} 1 & & 0 \\ & e^{-\lambda_2 t} & \\ 0 & & \ddots \; e^{-\lambda_n t} \end{bmatrix} U^T x_0$$

$$= \sum_{i=1}^{n} e^{-\lambda_i t} (u_i^T x_0) u_i$$

$$= \left(\mathbb{1}/\sqrt{n}^T x_0\right) \frac{\mathbb{1}}{\sqrt{n}} + e^{-\lambda_2 t} (u_2^T x_0) u_2$$

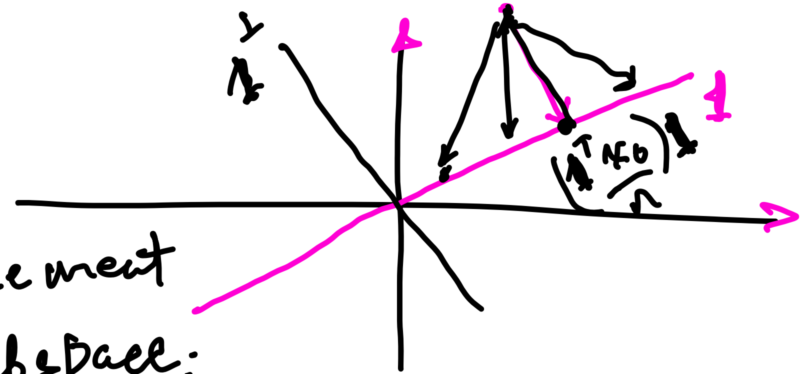
$$+ \cdots + e^{-\lambda_n t} (u_n^T x_0) u_n$$

- if $\lambda_2 > 0 \longrightarrow x(t) \rightarrow \left(\frac{\mathbb{1}^T}{\sqrt{n}} x_0\right) \frac{\mathbb{1}}{\sqrt{n}} = \frac{\mathbb{1}^T x_0}{n} \mathbb{1}$.

Recap:

$$x(t) \longrightarrow \underbrace{\mathrm{span}\{\mathbb{1}\}}_{}$$

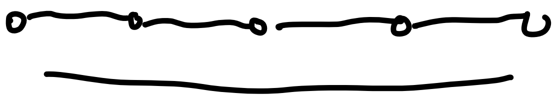
$\mathcal{A}$ = agreement subspace:

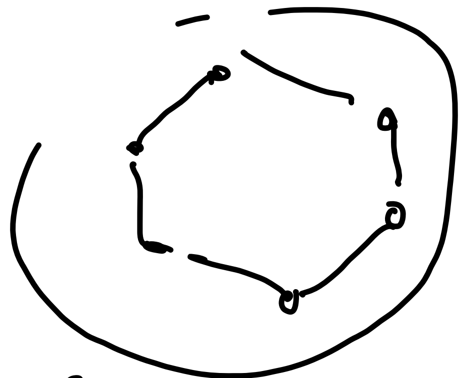


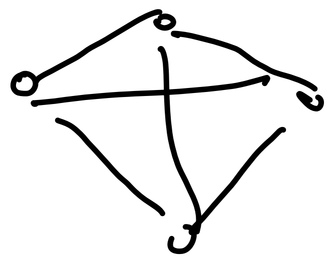
$\longrightarrow \mathbb{1}^T x(t)$?

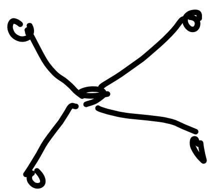
$$\frac{d}{dt}\, \mathbb{1}^T x(t) = -\underbrace{\mathbb{1}^T\, \mathcal{L}(G)}_{}\, x = 0$$

λ_2

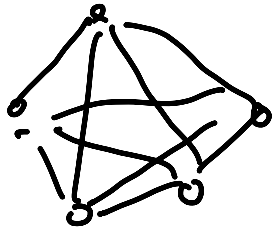
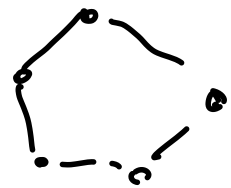








? $\mathcal{L}(K_n)$ $\qquad$ $\mathcal{L}(C_n)$ $\qquad$ $\mathcal{L}(P_n)$

regular: every node has the same deg.

$$\mathcal{L} = \Delta - A \leftarrow \text{adj.}$$

$$= kI - A$$

$$\mathcal{L}(K_n) = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots -1 \\ & & \cdots & n-1 \end{bmatrix}$$

$$= (n-1)I - \mathbf{1}\mathbf{1}^T + I$$

$$= nI - \mathbf{1}\mathbf{1}^T$$

spectrum: $0, \overbrace{n, n, \ldots, n}^{(n-1)}$

$$\mathcal{L}(K_n)v = \lambda v$$

$$\left(nI - \underbrace{\mathbb{1}\mathbb{1}^T\right)}_{\succeq 0} v = \underset{\overset{\uparrow}{n}}{?} \; v$$

$$e^{-nt}$$