Lecture 3-2

Jan. 19, 2024

Quote of the day:

"There's only one difference between art & science. In science, the universe is in control. In art, is in control you are."

Harry Kroto

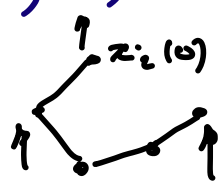
Today:

- Graph Spectra/consequences
- Directed/weighted consensus

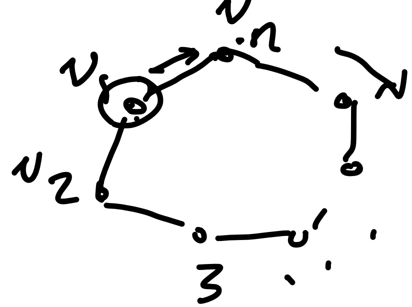
Last time:

spec of $\mathcal{L}(K_n)$: $0, \underbrace{n}_{\lambda_2}, \ldots, n$

$\Rightarrow$ spec of $A(K_n)$: $n-1, -1, \ldots, -1$



Spectra of Cycle graph: C_n



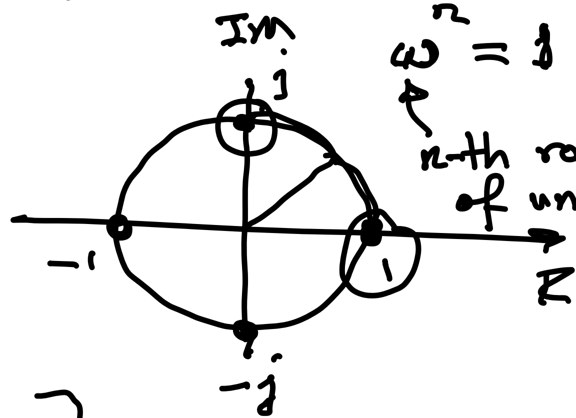
$\mathcal{L}(C_n) \sim A(C_n)$

2-regular

$$A(C_n)\, v = \lambda\, v$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ & & & \ddots & & \\ & & & & & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\begin{bmatrix} \omega^0 \\ \omega^1 \\ \omega^2 \\ \vdots \\ \omega^{n-1} \end{bmatrix}$$



$\omega^n = 1$

n-th root of unity

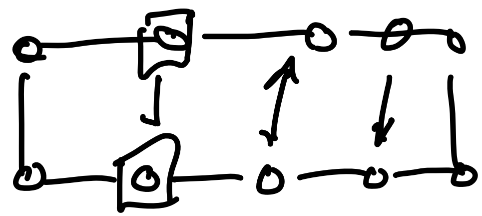
$$\omega = \cos\frac{2\pi k}{n} + j\sin\frac{2\pi k}{n} \qquad k = 0, \dots, n-1$$

$$= e^{j\frac{2\pi k}{n}}$$

$$\omega^{i-1} + \omega^{i+1} = \lambda\, \omega^i$$

$$\lambda = \bar{\omega} + \omega$$

C_{2n}

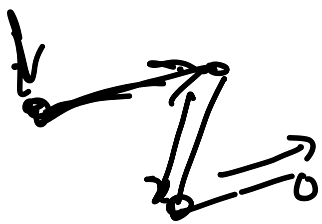


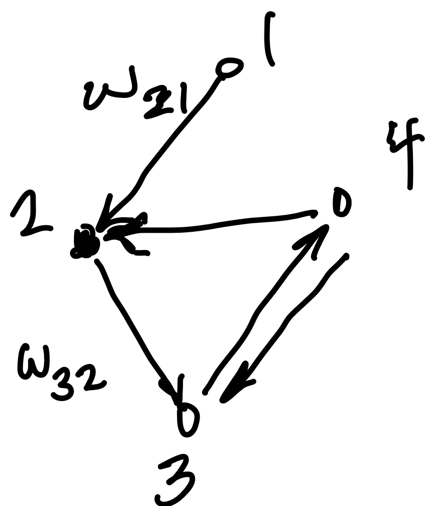
$$\lambda = 2\cos\frac{2\pi k}{n} \qquad k = 0, 1, \dots, n-1$$

$$\lambda(\mathcal{L}) \approx 2 - 2\cos\frac{2\pi k}{n} \qquad k = 0, 1, \dots, n-1$$

$$\begin{bmatrix} \omega_1^0 & \omega_2^0 & \cdots & \omega_n^0 \\ \vdots & \vdots & & \vdots \\ \omega_1^{n-1} & \omega_2^{n-1} & \cdots & \omega_n^{n-1} \end{bmatrix} \leftarrow \text{DFT matrix}$$

$$\lambda(P_n) = 2 - 2\cos\frac{\pi k}{n} \qquad k = 0, 1, \ldots, n-1$$





Directed graph

$$\dot{x}_1 = 0$$

$$+\omega_{24}(x_4 - x_2)$$

$$\dot{x}_2 = \omega_{21}(x_1 - x_2)$$

$$\dot{x}_3 = \omega_{32}(x_2 - x_3) + \omega_{34}(x_4 - x_3)$$

$$\dot{x}_4 = \omega_{43}(x_3 - x_4)$$

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ w_{21} & -(w_{21}+w_{24}) & 0 & w_{24} \\ & & \ddots & \\ & & & \end{bmatrix}}_{-\left(\text{in-degree Laplacian}\right)}$$

$$\underbrace{[A(D)]_{ji}}_{\text{in-deg adjacency}} = \begin{cases} w_{ji} & \text{if } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$i \longrightarrow j$$

$$\Delta(D) = \operatorname{diag}(A(D)\,\mathbb{1}) \longleftarrow \text{in deg. matrix}$$

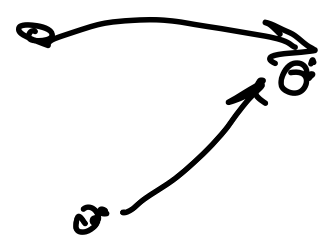
$$L_{in}(D) = \Delta(D) - A(D)$$

$$\dot{x} = \underbrace{-\mathcal{L}(\mathcal{D})}_{\text{not symmetric}}\, x \quad \leftarrow \text{directed consensus.}$$

$$\mathcal{L}(\mathcal{D})\,\mathbb{1} \overset{?}{=} 0$$

? convergence $\rightsquigarrow$ connectivity?

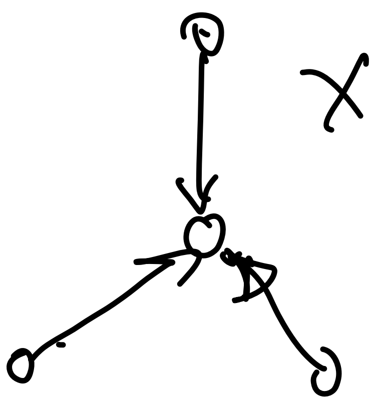
Roofed Outbranching

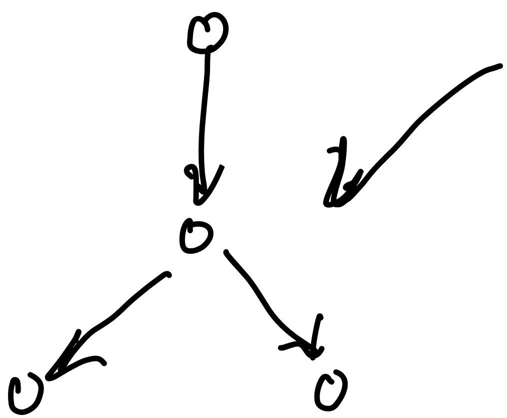


$$\mathcal{D} = (V, E)$$

D has a rooted out branching when

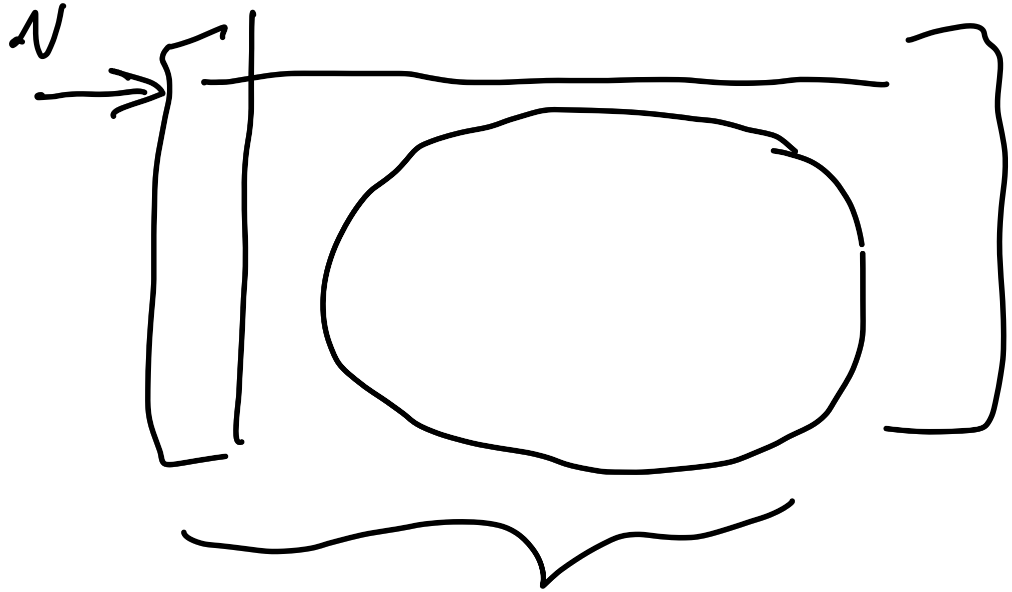
1) no directed cycles
2) $\exists\, v_r$ s.t. $\exists$ directed path from v_r to every other node





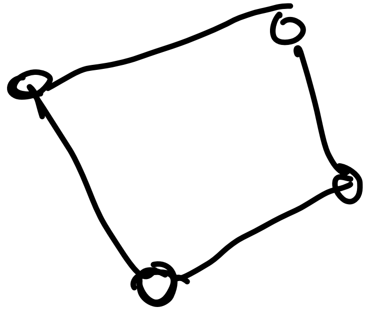
Kirchoff matrix tree theorem:

$\mathcal{L}(D)$



delete row/colm. corresponding to v. $\Rightarrow \mathcal{L}_v(D)$

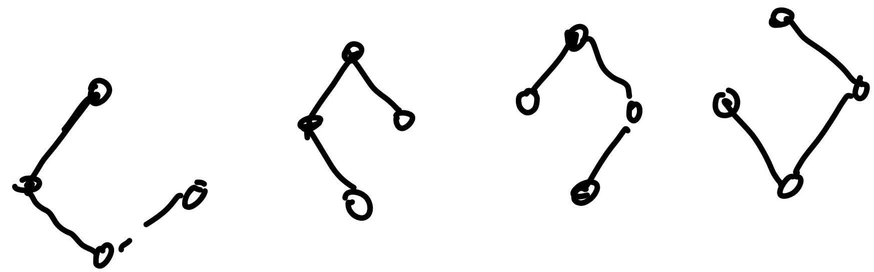
$$\det \mathcal{L}_v(D) = \sum_{T \in \mathcal{T}_v} \prod_{e \in T} w(e)$$

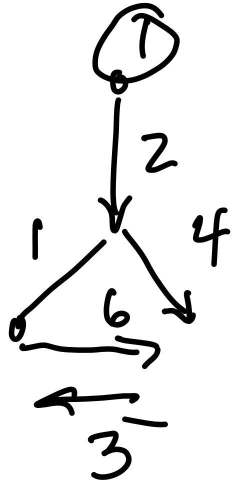


rooted outbranchings in D starting from v.

$$\mathcal{L}(C_4) = \begin{bmatrix} \cancel{2} & \cancel{-1} & \cancel{0} & \cancel{-1} \\ \cancel{-1} & 2 & -1 & 0 \\ \cancel{0} & -1 & 2 & -1 \\ \cancel{-1} & 0 & -1 & 2 \end{bmatrix}$$

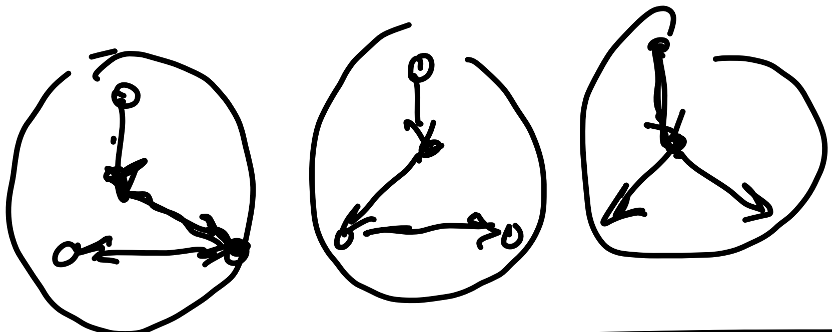
det $= 4$





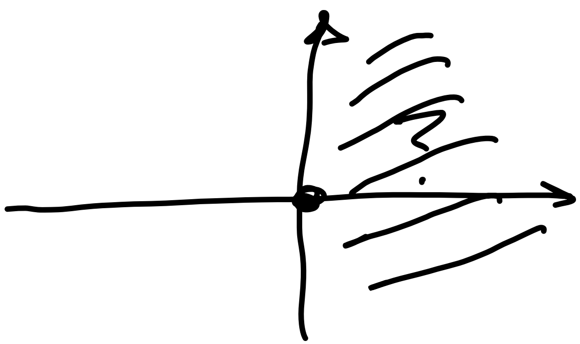
$$\mathcal{L}(D) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -1 & 4 & -3 \\ 0 & -4 & -6 & 10 \end{bmatrix}$$

$$= 44.$$

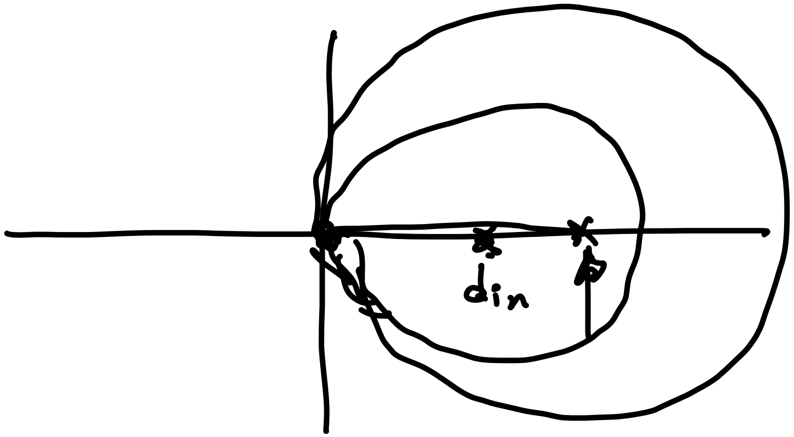


Thm: $\operatorname{rank} \mathcal{L}(D) = n-1$ iff D contains a rooted out branching

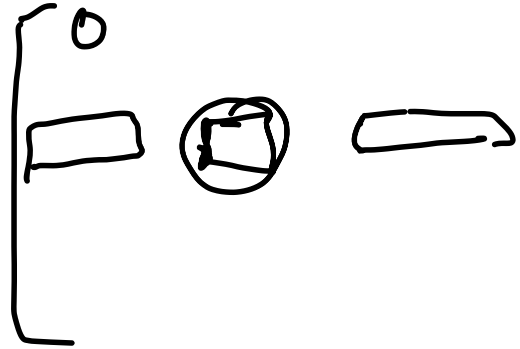
?

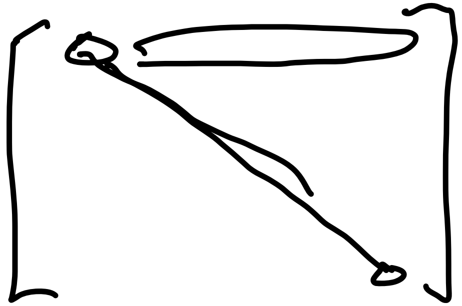


$$\dot{x} = -\mathcal{L}(D)\, x$$



$L(D) =$





Geršgorin disk Theorem.

$x(t) \longrightarrow a\mathbb{1}.$