Lecture 4-1

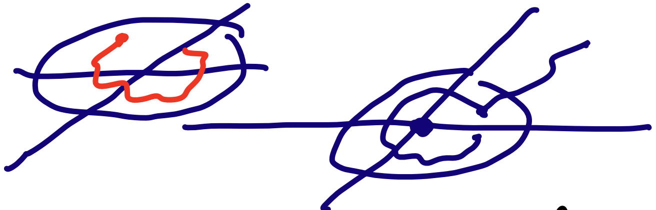
Lyapunov Analysis on Networks

Quote of the day

"Television is something the Russians invented to destroy American Education."

Erdős

Lyapunov (2^{nd} method) method is a very general & powerful machinery to check the stability of equilibria for dynamical systems described by ordinary differential equations of the form



$$\dot{x} = f(x) \qquad f: \mathbb{R}^n \to \mathbb{R}^n$$

typically assumed to be differentiable

An equilibrium for this dynamical system is any $\bar{x}$ s.t. $f(\bar{x}) = 0$, that if $x(\bar{t}) = \bar{x}$, then $x(t) = \bar{x}$ $\forall$ $t \geq \bar{t}$.

This idea can be extended to sets under the notion of invariance... $\Omega \subseteq \mathbb{R}^n$ is invariant for $f: \mathbb{R}^n \to \mathbb{R}^n$ if once $x(\bar{t}) \in \Omega$ for some $\bar{t}$, then $x(t) \in \Omega$ $\forall$ $t \geq \bar{t}$.

this is called forward invariance

Now, $\dot{x} = f(x)$ when initialized from some $x_0 \in \mathbb{R}^n$, can create a very complicated trajectory $x(t)$ — but if it happens that we can come up with some scalar function of x, a counter, a summarizing quantity, that qualitatively captures what the system is doing, then we will be grateful.

More formally a Lyapunov function for an equilibrium $\bar{x}$ of f, is a real-valued continuous function on some $\Omega \in \mathbb{R}^n$ containing $\bar{x}$ s.t: $\underbrace{V(\bar{x}) = 0}$, $V(x) > 0 \;\; \forall\, x \in \Omega, x \neq \bar{x}$

&

$$\dot{V}(x) = \nabla V^T f(x) \leq 0 \quad \forall\, x.$$

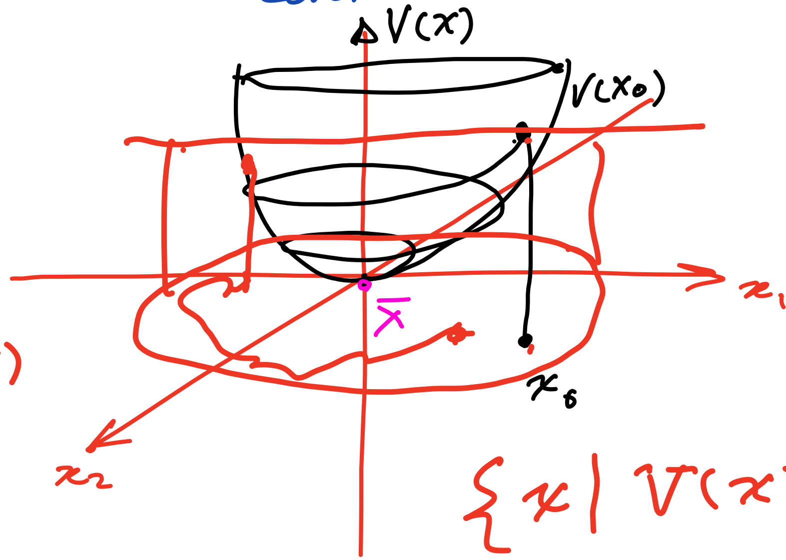
the value of V along any trajectory contained in Ω never increases.

we can also relax this so that $\bar{x}$ is the unique minimum point of V

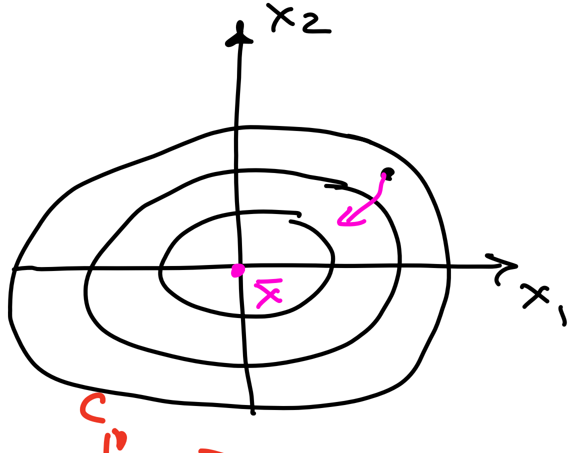
$$\frac{d}{dt} V(x) = \dot{V}$$

$$\rightsquigarrow \nabla V^T f(x)$$

$$V : \mathbb{R}^n \rightarrow \mathbb{R}$$



$$\{x \mid V(x) \leq V(x_0)\}$$



Thm: If $\bar{x}$ admits a Lyapunov function in some $B(\bar{x}, r)$ then $\bar{x}$ is stable. If $\dot{V}(x) < 0$ $\forall\, x \neq \bar{x}$, then $\bar{x}$ is asymptotically stable.

Asympotic stability means exactly that... $\bar{x}$ is stable & $x(t) \to \bar{x}$ when x is initialized in some neighborhood of $\dot{x}$. In order to get global asymptotic stability we also require radial unbounded of V, i.e.,

$$V(x) \to \infty \quad \text{when} \quad \|x\| \to \infty.$$

Note that if $\dot{V}(x) = 0$ for some $x \neq \bar{x}$, then asymptotic stability of $\bar{x}$ <u>does not</u> follow! But now the question is

where does $x(t)$ go ?!

La Salle's Invariance: Let

$$\Omega_s = \{ x \in \mathbb{R}^n \mid \underline{V(x) < s} \}$$

continuous gradients

be bounded & $\dot{V}(x) \le 0$.

Then $x(t) \to$ largest invariant set in $\{ x \in \Omega_s \mid \dot{V}(x) = 0 \}$.

Note that in this case you don't even need positivity of V!

we have to somehow ensure boundedness!

These results have a variety of extensions, including to switches systems.

If $\dot{x} = f_\sigma(x)$ & admits a (radially unbounded) common Lyapunov function, that is: $\begin{cases} \nabla V^T f_\sigma(x) < 0 & \forall x \neq 0 \ \ \forall \sigma \\ V(x) > 0 & \forall x \neq 0 \end{cases}$

then $x(t) \to 0$ (globally).

Similarly if the largest invariant set for all f_σ is the same & the solution remains bounded, then $x(t) \to$ largest invariant set contained in $\{x \mid \dot{V}(x) = 0\}$

$$\dot{x} = \begin{cases} A_1 x \\ A_2 x \end{cases}$$

$V(x) = x^T P x$? $\qquad$ P p.d.

$V(x) > 0 \quad \forall x \neq 0$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = (A_i x)^T P x + x^T P (A_i x)$$

$$= x^T (A_i^T P + P A_i) x$$

Okay - now we have this powerful machinery that we will apply to Networked system - in order to get a hang of this let us consider $\dot{x} = -\mathcal{L}(G)x$ & see if we can apply the Lyapunov setup. Well, what should our Lyapunov function be?

$\dot{x} =$

Let us try
$$V(x) = \frac{1}{2} x^T x$$

Then $\dot{V}(x) = \frac{1}{2}(\dot{x}^T x + x^T \dot{x}) = \frac{1}{2}(-x^T \mathcal{L}(G)x - x^T \mathcal{L}(G)x) = -x^T \mathcal{L}(G)x$

& if G is connected the largest invariant set contained in $\{x \in \mathbb{R}^n \mid \dot{V}(x) = 0\} = \text{span}\{\mathbb{1}\} = \mathcal{A}$ ← agreement set

So this is good but of course $V(x) \neq 0$ for $x \in \mathcal{A}$ if $\mathbb{1}^T x(0) \neq 0$.

Might be good to interpret what is $V(x)$ meaning?

if we consider $\mathcal{L}(K_n) = nI - \mathbb{1}\mathbb{1}^T$ then $x^T \mathcal{L}(K_n) x = n x^T x - (\mathbb{1}^T x)^2$

but $\frac{d}{dt}(\mathbb{1}^T x) = \mathbb{1}^T \dot{x} = \mathbb{1}^T(-\mathcal{L}(G)x) = 0$ so $\mathbb{1}^T x$ is a constant

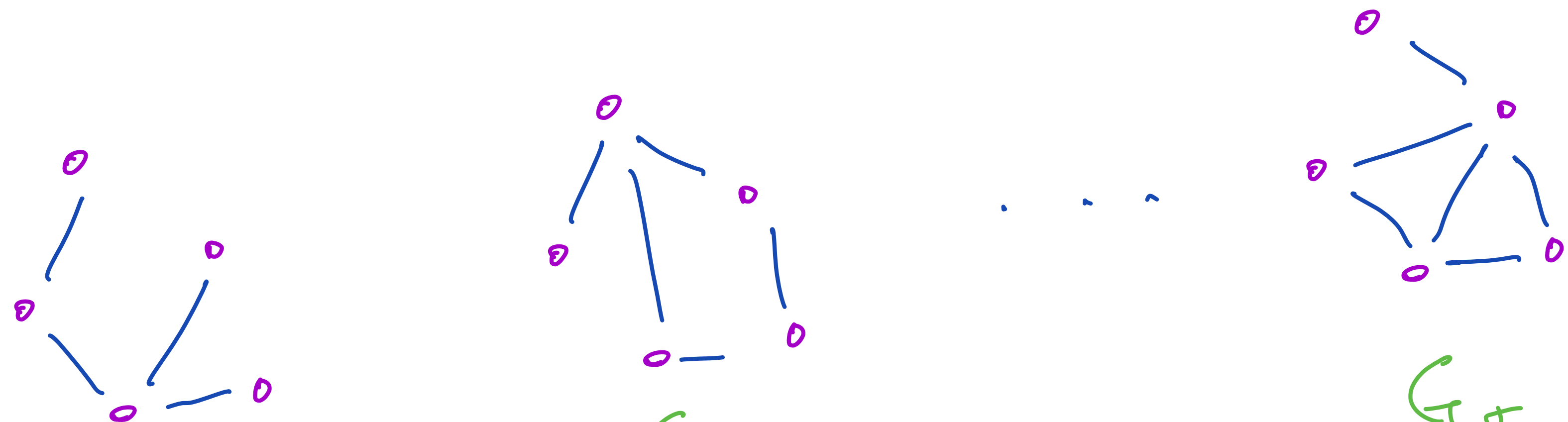
$$\rightsquigarrow \quad V(x) = \frac{1}{2} x^T x = \frac{1}{2n}\left(x^T \mathcal{L}(K_n) x + \underbrace{(\mathbb{1}^T x)^2}\right)$$

So x^* that minimize $V(x)$ is the same x for which $x^T \mathcal{L}(K_n) x = 0 \rightsquigarrow x \in \mathcal{A}$!

But we already know this! The power of Lyapunov is that if we have some "nonlinearity" Let us say that we have

$$\dot{x} = -\mathcal{L}(G_\sigma) x$$

where σ is the "mode" of the graph



G_{σ_1} G_{σ_2} . . . G_{σ_n}

Lyapunov analysis allows us to analyze consensus on switching & time varying networks!