Lecture 4-2

Jan 26, 2024

- Lyapunov analysis
- Planar unicycle (2D)
- connections to Kuramoto dynamics

Quote of the day

- "Look for what you notice but no one else sees."

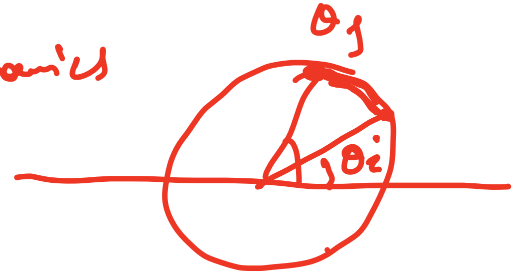
Rick Rubin

Let us consider

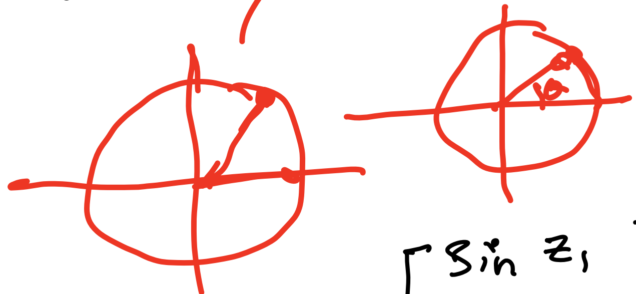
Kuramoto dynamics

$$\dot{\theta}_i = \sum_{i \sim j} \sin(\theta_j - \theta_i)$$

unicycle



This is called the Kuramoto dynamics. Let us consider



$$\underbrace{D(G)^T \theta}_{z} = \begin{bmatrix} \vdots \\ \theta_j - \theta_i \\ \vdots \end{bmatrix}$$

so $z = \begin{bmatrix} \vdots \\ \text{potential across each edge} \\ \vdots \end{bmatrix}$ Denote by $\sin(z) = \begin{bmatrix} \sin z_1 \\ \vdots \\ \sin z_m \end{bmatrix}$

$$\Rightarrow \dot{\theta} = -D(G) \sin z \Rightarrow D(G)^T \dot{\theta} = -D(G)^T D(G) \sin z$$

(G)

$$= D(G) D G^T$$

$$\Rightarrow \boxed{\dot{z} = - \mathcal{L}_e \sin z}$$

we call this the "edge laplacian"

Recall that

$$\mathcal{L}(G) = D(G)\, D(G)^T$$

$n \times n$

$$\mathcal{L}_e(G) = D(G)^T D(G)$$

$m \times m$

in subsequent discussion we assume that G is a spanning tree on n-nodes.

if G is a spanning tree, then $\mathcal{L}_e$ $n-1 \times n-1$

eigenvalues of AB & BA are the same except the zero eigenvalues

$\rightsquigarrow$ $\mathcal{L}_e(G)$ is positive def.

& min eigenvalue of $\mathcal{L}_e(G)$ is $\lambda_2(G)$!

$\mathbb{1}^T \cos z = \sum \cos(z_i) \rightsquigarrow \sum \sin z_i \, \dot{z}_i =$

Let us consider $V(z) = \mathbb{1}^T(\mathbb{1} - \cos z)$

p.d if G is a tree!

$$\Rightarrow \dot{V}(z) = \dot{z}^T \sin z - \sin z^T L_e \sin z$$

Note that $V(z) \geq 0$ & $V(z) = 0$ iff $\cos z = \mathbb{1}$

$z_i = z_j \quad \forall \, i,j \in E(G)$

Note that $V(z)$ is not radially unbounded & in fact $\sin z = 0$ for $z = \pi$ as well!

But this does imply that $z = 0$ is at least locally stable.

$$\Rightarrow \theta(t) \to \mathbb{A} \, !$$

Lyapunov theory & LaSalle's Invariance are very convenient to use to show asymptotic stability for networks that are dominated by diffusion, yet perturbed by "small" nonlinearities. For example, suppose we look at consensus on a tree graph,

$$\dot{x}_i = \sum_{j \sim i} (x_j - x_i) + \underbrace{f(x_j - x_i)}$$

a nonlinearity s.t.

$$\frac{f(x_j - x_i)}{\|x_i - x_j\|} \to 0$$

as $\|x_i - x_j\| \to 0$

Then we can rewrite this as

$$\dot{z} = -\mathcal{L}_e(G)\, z + g(z)$$

where $\mathcal{L}_e$ is the edge Laplacian of the tree graph (positive definite) &

$\frac{g(z)}{\|z\|} \to 0$ as $\|z\| \to 0$. In such a case, we can show that $\exists$ a neighborhood of $z = 0$ for which $V(z) = \frac{1}{2} z^T z$ is s.t. $\dot{V}(z) \leq 0$ $\forall \|z\| \leq d$

why is that the case? well

$$\dot V(z) = \dot z^T z = \left(-\mathcal{L}_e(G) z + g(z)\right)^T z$$

$$= \underbrace{-z^T \mathcal{L}_e(G) z}_{} + \underbrace{g(z)^T z}_{}$$

$$\le -\lambda_2 \|z\|^2 + g(z)^T z$$

if $\dfrac{g(z)}{\|z\|} = \rho(z)$ then $\|g(z)\| \le \rho(z)\,\|z\|$

$\rightarrow |g(z)^T z| \le \|g(z)\|\,\|z\|$

$\rightarrow \rho(z)\,\|z\|^2$ & $\rho(z) \to 0$ as $\|z\| \to 0$

~~so~~ since $\lambda_2 > \lambda_2 > 0$,

& $\dot V(z) \le \left(-\lambda_2 + \rho(z)\right)\|z\|^2$ & if $\rho(z) \to 0$ as $\|z\| \to 0$

$\exists\ \epsilon > 0$ s.t. $\forall\ \|z\|^2 \le \epsilon$, $-\lambda_2 + \rho(z) < 0$ &

Let us see how this approach works out for directed graphs (digraphs): let

$$\dot{x} = -\mathcal{L}(D)\, x$$

Recall that by construction $\mathcal{L}(D)\mathbb{1} = 0$ (in-degree Laplacian!)

if we consider something like $V(x) = x^T \mathcal{L}(K_n)\, x$

$$= n \underbrace{x^T x} + \underbrace{(\mathbb{1}^T x)^2}_{x^T \underbrace{\mathbb{1}\mathbb{1}^T}_{J} x}$$

then

$$\dot{V}(x) = \underbrace{n\left(\dot{x}^T x + x^T \dot{x}\right)}_{n\left(-x^T \mathcal{L}(D)^T x - x^T \mathcal{L}(D) x\right)} + \underbrace{\dot{x}^T J x + x^T J \dot{x}}_{-x^T \mathcal{L}(D)^T J x - x^T J L(D) x}$$

$$= -n\, x^T \left(\mathcal{L}(D) + \mathcal{L}(D)^T\right) x - x^T \left(\mathcal{L}(D)^T J + J \mathcal{L}(D)\right) x$$

? what does $\mathcal{L}(D) + \mathcal{L}(D)^T$ look like

? what is $\mathbb{1}^T \mathcal{L}(D)$

$$\mathbb{1}^T \mathcal{L}(D) = 0 \quad \Rightarrow \quad \begin{bmatrix} \deg_{in}(v_1) & -1 & \cdots \\ & \ddots & \\ & & \ddots \end{bmatrix}$$

indegree = out degree

The digraph is balanced if $\forall v_i$, $\deg_{in}(v_i) = \deg_{out}(v_i)$

In this case $\mathbb{1}^T \mathcal{L}(D) = 0 \Rightarrow \frac{d}{dt}(\mathbb{1}^T x) = -\mathbb{1}^T \mathcal{L}(D) x = 0$

$\rightsquigarrow$ the sum of the states is a conserved quantity

In this case

$$\dot{V} = -\frac{k}{2} x^T \{ \mathcal{L}(D) + \mathcal{L}(D)^T \} x$$

if D is balanced & weakly connected $\rightsquigarrow$ strongly connected!

in this case $\mathcal{L}(D) + \mathcal{L}(D)^T \succeq 0 \rightarrow$ LaSalle's Principle

extendible to switching case as well!

So far we have looked at the consensus dynamics

$$\underbrace{\dot{x} = -\mathcal{L}(\mathcal{G})x}_{\substack{\text{undirected/connected}\\ \text{networks; } x_i \in \mathbb{R}}} \quad \& \quad \underbrace{\dot{x} = -\mathcal{L}(\mathcal{D})x}_{\substack{\text{directed/containing}\\ \text{a rooted out-branching}\\ \text{network; } x_i \in \mathbb{R}}}$$

just a note that using Kronecker products we can have the consensus protocol for $x_i \in \mathbb{R}^d$ – more on this next week.

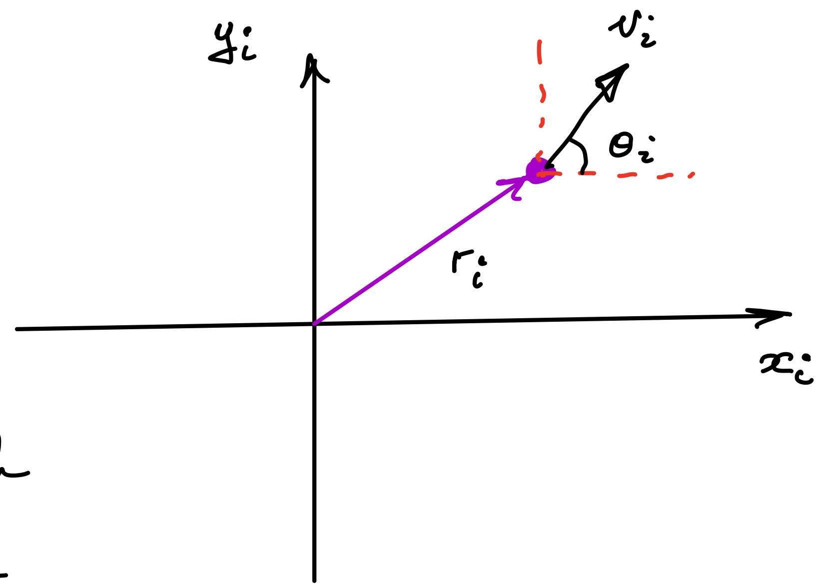
Typically, when we think of formation control, we think of coordination in the state of the agents, e.g., position, velocity, etc. Today I want to explore how coordination can involve angular states – one of the nice vehicular models for this purpose is the so-called unicycle model... yes, it is not a bicycle :-

So what is the unicycle model? Consider the vehicle model with a constant speed (this can be generalized of course). The "control" for this constant speed vehicle is the heading; in pictures it looks like this:

the control is of the form

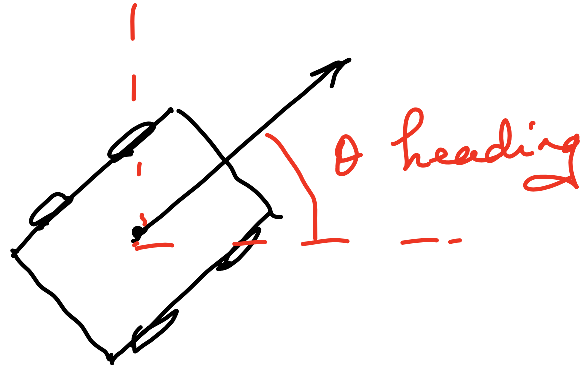
$$\dot{\theta}_i = u_i$$

angular rate ω_i



So what would a formation of such vehicles look like? Well it could be that eventually all heading angles are the same, e.g., heading synchronization. Or maybe we want all of them to all angles that are uniformly distributed between zero & 2π. Note that for unicycles it is unreasonable to think of position control, since the speed is constant. So for constant speed unicycles we think of phase synchronization.

Okay - so we have a 2-dimensional dynamics & we care about the phases ... seems like a perfect candidate to map the dynamics to $\underbrace{\text{complex numbers}}$!

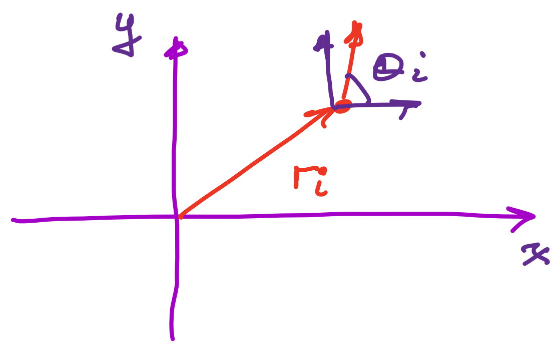


denote this by $\mathbb{C}$.

This is how it goes... we let

$$r_i(t) = x_i + j\, y_i(t)$$

$j \hookrightarrow \sqrt{-1}$



speed is constant so:

$$v_i = \left(\dot{x}_i^2 + \dot{y}_i^2\right)^{1/2}$$

v_i: speed

In fact

$$\left\{\begin{array}{l} \dot{x}_i = v_i \cos\theta_i \\ \dot{y}_i = v_i \sin\theta_i \\ \dot{\theta}_i = u_i \end{array}\right. \quad \Rightarrow \quad \begin{aligned} \dot{r}_i &= \dot{x}_i + j\,\dot{y}_i \\ &= v_i \cos\theta_i + j\, v_i \sin\theta_i \\ &= v_i\, e^{j\theta_i} \end{aligned}$$

salute to Euler!

we let $v_i = 1$ for the rest of our discussion

so the dynamics for each agent can be written as

$$\dot{r}_i = e^{j\theta_i} \qquad \dot{\theta}_i = u_i$$

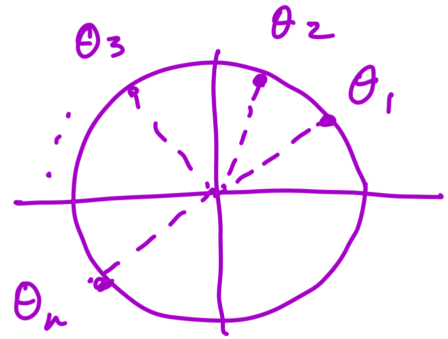
this is distinct for example from

$$\dot{r}_i = v_i \qquad \dot{v}_i = u_i$$

both magnitude/direction!

Now we monitor the evolution of θ_i's. In fact, we consider n - unicycles running around with constant speed with heading θ_i's & our group state vector look like:

$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$$



we are interested in local interaction rules that lead to phase synchronization, i.e., $\theta_1 = \theta_2 = \dots = \theta_n$.

we will think of the state of the agents as $e^{j\theta_i}$

Notation: $e^{j\theta} = \begin{bmatrix} e^{j\theta_1} \\ \vdots \\ e^{j\theta_n} \end{bmatrix}$ $\qquad (e^{j\theta})^* = \begin{bmatrix} e^{-j\theta_1} & \dots & e^{-j\theta_n} \end{bmatrix}$

complex conjugate transpose

If you recall, for consensus dynamics $V(x) = \frac{1}{2}\|x\|^2$ provided the Lyapunov "counter" that was related to $\frac{1}{2}x^T \mathcal{L}(K_n)\, x$ ~ the total disagreement amongst the nodes. Let us see what

$$\left(e^{j\theta}\right)^* \mathcal{L}(K_n)\, e^{j\theta}$$

looks like?

well:

$$\left(e^{j\theta}\right)^* (n\mathbb{I} - \mathbb{1}\mathbb{1}^T)\left(e^{j\theta}\right) = n\left(e^{j\theta}\right)^*\left(e^{j\theta}\right) - \left(e^{j\theta}\right)^* \mathbb{1}\mathbb{1}^T \left(e^{j\theta}\right)$$

$$= n^2 - \left(e^{j\theta}\right)^* \mathbb{1}\mathbb{1}^T e^{j\theta}$$

$$e^{-j\theta_i} e^{j\theta_i} = 1$$

Let us consider the potential

$$\mathcal{U}(\theta) = \frac{1}{2n}\left(e^{j\theta}\right)^* \mathbb{1}\mathbb{1}^T e^{j\theta}$$

for synchronization we want to maximize this potential!