Lecture 5-1

- Unicycle formations
- Formation Control

Quote of the day

"Look for what you notice but no one else sees."

Rick Rubin

Links

https://youtu.be/eakKfY5aHmY

https://www.robotarium.gatech.edu/get_started

Quote of the day:

"This book tries to explain how mind works. How can intelligence emerge from nonintelligence? To answer that, we show that you can build a mind from many little parts, each mindless by itself."

Marvin Minsky – prologue to "The Society of Mind"

If you recall, for consensus dynamics $V(x) = \frac{1}{2}\|x\|^2$ provided the Lyapunov "counter" that was related to $\frac{1}{2} x^T \mathcal{L}(K_n) x \sim$ the total disagreement amongst the nodes. Let us see what

$$\left(e^{j\theta}\right)^* \mathcal{L}(K_n)\, e^{j\theta}$$ looks like?

well:

$$\left(e^{j\theta}\right)^* \left(n I - \mathbb{1}\mathbb{1}^T\right)\left(e^{j\theta}\right) = n\left(e^{j\theta}\right)^*\left(e^{j\theta}\right) - \left(e^{j\theta}\right)^* \mathbb{1}\mathbb{1}^T \left(e^{j\theta}\right)$$

$$= n^2 - \left(e^{j\theta}\right)^* \mathbb{1}\mathbb{1}^T e^{j\theta}$$

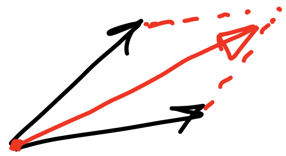
$$e^{-j\theta_i} e^{j\theta_i} = 1$$

Let us consider the potential

$$\mathcal{U}(\theta) = \frac{1}{2n}\left(e^{j\theta}\right)^* \mathbb{1}\mathbb{1}^T e^{j\theta}$$

for synchronization we want to maximize this potential!

What is $\mathbb{1}^T e^{j\theta} = \sum_i e^{j\theta_i}$ & $U(\theta) = \frac{1}{2n} \| \sum e^{j\theta_i} \|^2$



if $\theta_1 = \theta_2$ then $\| e^{j\theta_1} + e^{j\theta_2} \|$ is bigger

In fact if $\theta_1 = -\theta_2$ the $\| e^{j\theta_1} + e^{j\theta_2} \| = 0$!

The minimum occurs when $\mathbb{1}^T e^{j\theta} = 0 \longrightarrow$ we call this the balanced state

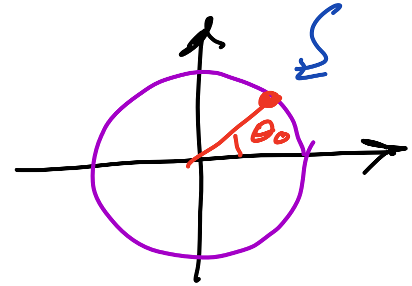
Note that

$$\frac{1}{n} \sum \dot{r}_i = \text{velocity of the geometric center}$$

$$= \frac{d}{dt}\left(\frac{1}{n} \sum r_i\right) = \frac{1}{n} \mathbb{1}^T e^{j\theta}$$

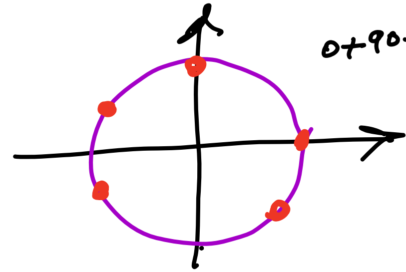
if we want to maximize a function we follow the gradient $\nabla U(\theta)$. The maximum occurs when

all phases



$$e^{j\theta} = \begin{bmatrix} e^{j\theta_0} \\ e^{j\theta_0} \\ \vdots \\ e^{j\theta_0} \end{bmatrix} = e^{j\theta_0}\,\mathbb{1}$$

& minimum occurs when $\mathbb{1}^T e^{j\theta} = 0$



0+90+120+195 +265=0°!

Okay – let us see how this works out:

we let $u_i = -k\,\nabla_i U(\theta)$ ← gradient w.r.t. ith coordinate

↳ if $K>0$ ⇝ move towards minimum → balanced

$K<0$ ⇝ move towards maximum → synchronization

Let us see what this is:

$$u_i = \frac{-k}{2n} \nabla_i \left(\sum e^{-j\theta_i} \right) \left(\sum e^{j\theta_i} \right)$$

$$= \frac{-k}{2n} \frac{d}{d\theta_i} \left\{ \left(\sum e^{-j\theta_i} \right) \left(\sum e^{j\theta_i} \right) \right\}$$

$$= \frac{-k}{2n} \frac{d}{d\theta_i} \left\{ e^{-j\theta_i} \left(\sum e^{j\theta_i} \right) \right\} + e^{j\theta_i} \left(\sum e^{-j\theta_i} \right) \Big\}$$

$$= \frac{-k}{2n} \frac{d}{d\theta_i} \Big\{ e^{j(\theta_1 - \theta_i)} + e^{j(\theta_2 - \theta_i)} + \cdots + e^{j(\theta_n - \theta_i)}$$

$$+ e^{-j(\theta_1 - \theta_i)} + \cdots + e^{-j(\theta_n - \theta_i)} \Big\}$$

$$= \frac{-k}{2n} \Big\{ -j \left(e^{j(\theta_1 - \theta_i)} + \cdots e^{j(\theta_n - \theta_i)} \right.$$

$$\left. + j \left(e^{-j(\theta_1 - \theta_i)} + \cdots + e^{-j(\theta_n - \theta_i)} \right. \right\}$$

$$= \frac{-K}{2n} \left\{ -j \left(\sum_{k} e^{j(\theta_k - \theta_i)} - e^{-j(\theta_k - \theta_i)} \right) \right\}$$

$$\approx \frac{-K}{n} \sum_{k} \sin(\theta_k - \theta_i)$$

← guess what? Kuramoto model!

What this means is that the stability of the unicycle phase synchronization is essentially the stability analysis for synchronization of $\theta_1 = \theta_2 = \dots = \theta_n$ for the Kuramoto dynamics. In this model we have

$K > 0 \rightsquigarrow$ min. of $U(\theta) \rightsquigarrow$ balanced

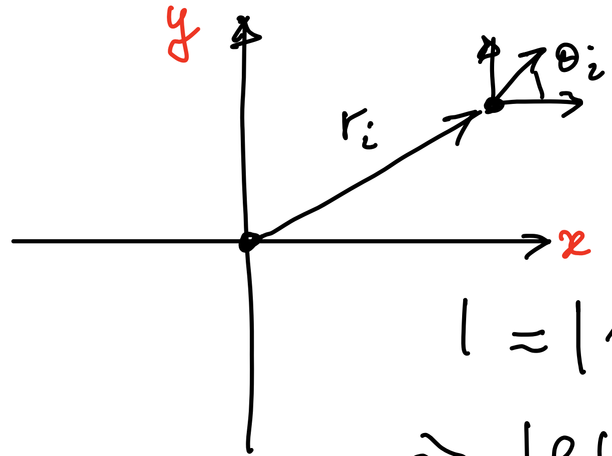
$K < 0 \rightsquigarrow$ max of $U(\theta) \rightsquigarrow$ synchrony

Every time you have a potential that you want to maximize or minimize by following the gradient (up or down!) you have to worry about other critical points where $\nabla f = 0$ but you are not necessary minimizing or maximizing the function → for the potential

$$\| \mathbb{1}^T e^{j\theta} \|^2$$

there are other critical points but they turn out to be unstable.

Another point about phase synchronization is that it is unclear if the unicycles will end up orbiting the same center eventually. This does depend on their initial conditions.



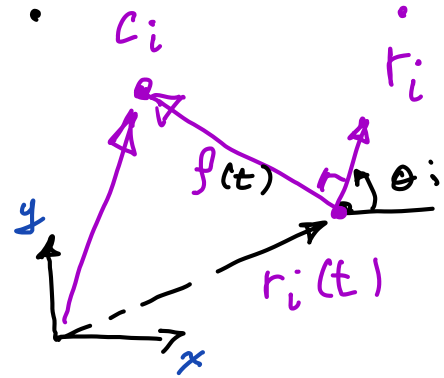
where is the center of the i-node orbit?

$$|= |v| = |\rho(t)|\,|\omega_0|$$

$$\Rightarrow |\rho(t)| = \frac{1}{|\omega_0|}$$

$$\rho(t) = \frac{j}{\omega_0} e^{j\theta_i}$$

$$\dot{r}_i = e^{j\theta_i}$$



$$c_i(t) = r_i(t) + \frac{j e^{j\theta_i}}{|\omega_0|}$$

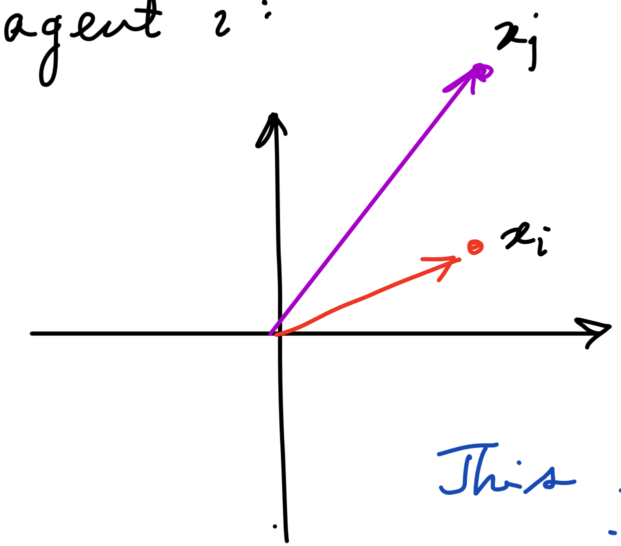
The trick for $\left\{\begin{array}{l}\text{phase synchronization} \\ \& \\ \text{synchronization of the center}\end{array}\right\}$ ↪ when $K<0$ & $\omega_0 \neq 0$

turns out to be realizable by introducing a new variable $q_i = -j\, c_i\, \omega_0 = -j\omega_0 r_i + e^{j\theta}$

& then following the gradient of $S(q) = \frac{1}{2} q^* \mathcal{L}(G) q$!

So far we have look at agreement/consensus on undirected & directed networks, highlighting the role of connectivity/rooted out branchings in convergence of distributed algorithms. We explored some spectral properties of graphs, & then looked at Lyapunov/La Salle's framework for analyzing synchronization phenomena. These algorithms have proven to be extremely useful to reason about behaviors emergent from local interactions. Hower at the surface their applicability seems to be limited to say agreement/synchronization/balanced configurations. So a natural question is whether they can be used for formation control ... What is a formation?

A formation is a shape or configuration maintained over some interval. So the first question deals with formation specification. For example a formation can be specified by a set of distances: let $x_i \in \mathbb{R}^3$ be the position of agent i:



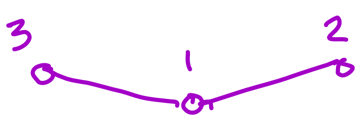
$$d_{ij} = \| x_i - x_j \|$$

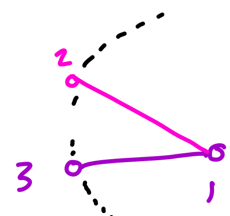
$$\mathcal{F}_d = \{ d_{ij} \mid d_{ij} > 0 \text{ for all } i,j \}$$

formation specification using distances.

This is certainly a specification but we realize that it might be too loose of a specification.

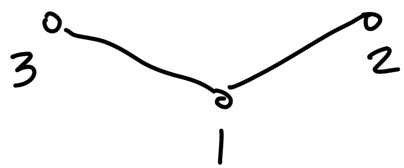
Example: $\mathcal{F}_D = \{ x_i \in \mathbb{R}^2 \mid d_{12} = d_{13} = 1 \}$





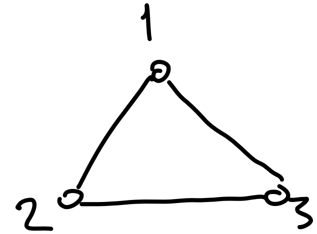
Such a specification might prove too loose.

<u>Def:</u> A formation (distance) specification is called rigid if it specifies the formation up to translation & rotation. For example:



not rigid

$\{d_{13}=d_{12}=1\}$



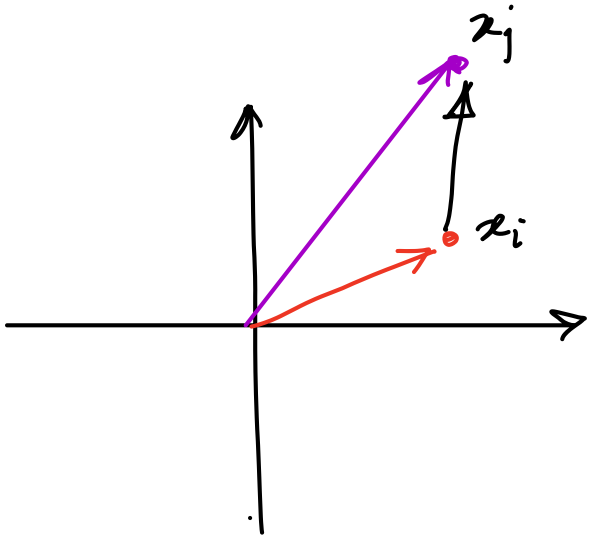
rigid

$F=\{d_{12}=d_{23}=d_{13}=1\}$

More generally if we have a metric space $\mathcal{M}$ & $x_i \in \mathcal{M}$ we can specify the distance formation via $d_{ij}=\rho(x_i,x_j)$

metric on $\mathcal{M}$.

Another type of specification, perhaps more natural, but using more information, is via relative states, e.g.,



$$z_{ij} = \underbrace{x_j - x_i} \quad \text{for } i,j \in V(G), \; x_i, x_j \in \mathcal{X}$$

relative state ... might be better to use the right notion of "difference" depending on $\mathcal{X}$.

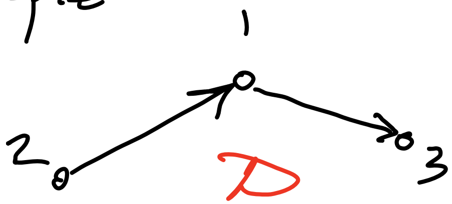
Observation: since $x_j - x_k = (x_j - x_m) + (x_m - x_k)$ then by specifying the relative state information on a spanning tree we have completely specified the formation.

Note that we can write:

$$z_{ref} = D(\mathcal{D})^T x$$

directed graph used for specification

For example



if $x_{21} = x_1 - x_2 \qquad \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$x_{13} = x_3 - x_1 \qquad \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$D(\mathcal{D}) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \rightsquigarrow D(\mathcal{D})^T = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow D(\mathcal{D})^T x = \begin{bmatrix} x_1 - x_2 \\ x_3 - x_1 \end{bmatrix}$$

if $x_i \in \mathbb{R}^d$ then

$$z = (D(\mathcal{D})^T \otimes I)\, x$$

A relative state specification on $V(G)$ is any weakly connected digraph. We denote an RSS by $\mathcal{Z}_{ref}$.

Okay ... let us see how this all work out for $\dot{x}_i = u_i$

Say we have specified z_{ref} & $\underbrace{z(t)}_{\text{the relative state at time } t} = D(\mathcal{D})^T x(t)$

Then

$$\text{error}(t) = e(t) = z_{ref} - z(t)$$

$$\Rightarrow \dot{e} = -\dot{z} = -D(\mathcal{D})^T \dot{x} = -D(\mathcal{D})^T u$$

For this analysis it is assumed that z_{ref} is constant how can this be changed if not?

then if we let

$$u(t) = k\, D(\mathcal{D})\, e$$

we realize that

$$\dot{e} = -k \underbrace{D(\mathcal{D})^T D(\mathcal{D})}_{\mathcal{L}_e(\mathcal{D})} e$$

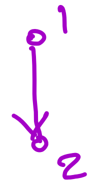
$$\Rightarrow \boxed{\dot{e} = -k\, \mathcal{L}_e(\mathcal{D})}$$

guess what! edge Laplacian!

if $\mathcal{D}$ is a spanning tree the $\mathcal{L}_e$ has positive eigenvalues & $e(t) \to 0$!

But what is $u(t) = k D(\mathcal{D})\, e$?

$$z_{ref} - z = D(\mathcal{D})^T x_r - D(\mathcal{D})^T x$$



$$z_{12} = x_2 - x_1$$

$$\rightsquigarrow \; u(t) = k\, D(\mathcal{D})\, D(\mathcal{D})^T (x_r - x)$$

$$= k\, \mathcal{L}(\tilde{\mathcal{D}})\,(x_r - x) = -k\,\mathcal{L}(\mathcal{G})\,x + k\,\mathcal{L}(\mathcal{G})\,x_r$$

the disoriented graph associated w/ RSS

$$= -k\,\mathcal{L}(\tilde{\mathcal{D}})\,x + k\, D(\mathcal{D})\underbrace{D(\mathcal{D})^T x_r}_{z_{ref}}$$

$$= \underbrace{-k\,\mathcal{L}(\tilde{\mathcal{D}})}_{A}\, x + \underbrace{k\, D(\mathcal{D})}_{B}\, z_{ref}$$